

Ultrametric Cantor sets and Origin of Anomalous Diffusion

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Abstract

The anomalous mean square fluctuations are shown to arise naturally from the ordinary diffusion equation interpreted scale invariantly in a formalism endowing real numbers with a nonarchimedean multiplicative structure. A variable t approaching 0 linearly in the ordinary analysis is shown to enjoy instead a sublinear $t \log t^{-1}$ flow in the presence of this scale invariant structure. Diffusion on an ultrametric Cantor set is also generically subdiffusive with the above sublinear mean square deviation. The present study seems to offer a new interpretation of a possible emergence of complex patterns from an apparently simple system.

Key Words: Anomalous diffusion, Cantor set, Scale invariant, Nonarchimedean

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1. Introduction

Anomalous diffusion is known to occur ubiquitously in diverse complex systems enjoying fine “structure with variations” [1] such as in disordered or fractal media, thus developing fat tailed, broad distributions and/or long range spatio-temporal correlations [2, 3]. The hallmark of such a diffusion process is the occurrence of an anomalous law for the mean square displacement (fluctuation), viz., $\langle \Delta x^2(t) \rangle = t^\nu$, with $\nu \neq 1$. Sub-diffusive ($\nu < 1$) behaviour is usually predominant in disordered systems, for instance, in spin glasses, amorphous semiconductors, lipid bilayers, living cells, transport in fractal sets and many others where a broad (fat tailed) distribution for local trapping times of the diffusive test particle gradually build up [2]. Super-diffusion ($\nu > 1$), on the other hand, may arise from long range correlations in velocity fields of turbulent flows, Levy flights and so on [3]. The emergence of nonlinear growth of the mean square fluctuation clearly reflects a possible violation of the gaussian central limit theorem in the underlying random walk processes [4]. An important, still unsolved, problem is to look for a generic (universal) mechanism for the emergence of anomalous diffusion in such diverse phenomena. In the present paper, we offer one such dynamical principle that might be at play at the heart of complex systems. We show that the anomalous mean square fluctuations can arise naturally from the ordinary diffusion equation interpreted scale invariantly in a formalism endowing real numbers with a nonarchimedean multiplicative structure. A variable t approaching 0 linearly in the ordinary analysis is shown to enjoy a sublinear $t \log t^{-1}$ flow in the presence of this scale invariant structure. Diffusion on an ultrametric Cantor set is also generically subdiffusive with the above seemingly universal sublinear mean square deviation. The present study seems to offer a new interpretation of a possible emergence of complex patterns from an apparently simple system. This, in turn, appears to realize the philosophical musings, “Nature can produce complex structures even in simple situations, and can obey simple laws even in complex situation [1].”

There is already a vast body of studies on anomalous diffusion and its origin that are available in literature [2, 3]. Even at this back ground, the present investigation aims at offering a potentially new insight into the actual mechanism of the dynamics of an anomalous motion. As it is well known, there are actually two distinct types of motion observed in Nature: smooth, regular motion, like the Newtonian (two body) planetary motion, and random, highly irregular motion, as in the Brownian motion of a fine pollen particle in a liquid at rest [3]. A smooth motion, at least on a moderate time scale, is expected to be predictable and so are deterministic in nature, when Brownian type motion requires statistical (stochastic) methods. Of course, the deterministic chaos falls in between, and several authors, for instance, Ref. [5], discussed the problem of offering a dynamical interpretation for the Brownian like motion based on the deterministic Hamiltonian models in the phase space. At a more elementary level, on the other hand, the classical Brownian motion can, in fact, be considered to enjoy a bit of a deterministic flavour as the relevant gaussian probability distribution (transition probability/ propagator) is known to follow

the linear homogeneous diffusion equation

$$\frac{\partial W}{\partial t} = \frac{\partial^2 W}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0 \quad (1)$$

where we choose the diffusion constant to be unity. Accordingly, variations of macroscopic variables such as the number of diffusive particles, concentration and similar other quantities are governed effectively by a smooth deterministic law. Mathematically, Brownian motion is a process realized in a homogeneous smooth manifold where Taylor's expansion and other relevant analytic function theoretic resources are available. Moreover, the probability density is also a smooth function having finite mean and variance, so that the universality of the central limit theorem drives the force law to be smooth (for another explanation, see [5]).

On the otherhand, if the underlying space of diffusion has a manifestly disordered, fractal structure, the above gaussian, and/or the function theoretic smoothness is generally lost (see for instance [6, 7, 8, 9, 10, 2]). All these studies tried to offer precise mathematical justifications leading to anomalous mean square variations in such fractal spaces [11]. It follows also that the emergence of a *smooth* effective deterministic law (in the sense of differentiable functions in a Euclidean space) at the macroscopic scale is generally lost because of the inherent loss of smoothness in the random process [12].

Over the past few years we have been developing a formalism of a scale invariant analysis [13, 14] aiming at integrating the framework of the standard analysis on, for instance, the Euclidean spaces and those on a fractal like space. Another motivation was to investigate *if a seemingly smooth deterministic evolution, even in the absence of any external influences, may lead naturally to a complex pattern over an asymptotically long (or short) time scale*. In the following subsection, we present a short summary of the basic ingredients of our approach [13, 14, 16, 17] that will be necessary for the subsequent analysis. However, most of the derivations are presented first time here offering better insights as compared to the original derivations. In the latter two sections, we present new results on Cantor sets and diffusion on a scale invariant real line and Cantor sets respectively. It will become clear that the presence of *dynamically* generated ultrametric Cantor sets at infinitesimally small finer scales in the real line would force a linear diffusive process to evolve anomalously over infinitely long time scales, that are available naturally in the scale invariant formalism.

1.1. Formalism

We construct a scale invariant, nonarchimedean extension of the real number system R accommodating the so called generalized solutions of the scale free Cauchy problem

$$t \frac{dx}{dt} = x, \quad x(1) = 1 \quad (2)$$

in the form

$$X_{\pm}(t) = t\tilde{X}_{\pm}(\tilde{t}) \quad (3)$$

where $\tilde{X}_\pm = \tilde{t}^{\mp v(\tilde{t})}$, $\tilde{t} = t/\epsilon$, and v satisfies the self similar replica of an inverted (2), viz,

$$\tau \frac{dv}{d\tau} = -v \quad (4)$$

in the logarithmic variable $\tau = \log \tilde{t}$, for an arbitrarily small ϵ . Notice that $X_\pm(1) = 1$ as one must set $\epsilon = 1$ and v is finite. Clearly, $v \equiv 0$ in the ordinary analysis on the real line. However, in an extended nonarchimedean model \mathbf{R} of the real number system R accommodating *nontrivial infinitesimals* in the neighbourhood of zero, this class of solutions becomes meaningful (for a justification see [14]). Notice that the ansatz (3) tells that each element $t \in R$ splits into pairs of infinitesimally close neighbours belonging respectively to the right and left neighbourhood of $t \in \mathbf{R}$. The scale free component \tilde{X} (for simplicity of notations, we drop temporarily subscripts \pm) of the generalized solution now is realized as a locally constant function (LCF) defined by

$$t \frac{d\tilde{X}}{dt} = 0 \quad (5)$$

As a consequence, $\tilde{t} = \tilde{t}(t) \in (0, 1)$ may, in fact, be *any* continuously first differentiable function of t , which is a consequence of the *reparametrisation invariance* of a locally constant function in an ultrametric space (this aspect of a LCF is discussed in more detail in [16]).

Now to construct a self consistent ultrametric extension suitable for our purpose, let us introduce scale invariant infinitesimals via a more refined evaluation of the limit $t \rightarrow 0^+$ in R . Notice that as $t \rightarrow 0^+$, there exists $\epsilon > 0$ such that $0 < \epsilon < t$ and one may like to identify zero (0) with the closed interval $I_\epsilon = [-\epsilon, \epsilon]$ at the *scale* (i.e. accuracy level in a computation) ϵ . Ordinarily, I_ϵ is a connected line segment, which shrinks to the singleton $\{0\}$ as $\epsilon \rightarrow 0^+$, so as to reproduce the infinitely accurate, *exactly determinable* ordinary real numbers.

Let us now present yet another *nontrivial mode* realizing the limiting motion $t \rightarrow 0^+$. This mode is nonlinear, as it is defined via *inversions*, rather than simply by the linear translations, that is available *uniquely* in the standard analysis on R . In the presence of *infinitesimals*, the present nonlinear mode may, nevertheless, assume significance. In the following we give a construction of an *explicitly defined infinitesimal*, without requiring the model theoretic set up of Robinson's nonstandard analysis. *Our formalism may be considered to be a new, independent realization of infinitesimals residing originally in an ultrametric space, but, nevertheless, inducing real valued "infinitesimal corrections" to the ordinary real variable* [18]. A real number, as it were, assumes a *deformed value* because of a possible nontrivial *motion* close to 0. Fix $\epsilon = \epsilon_0$ and let $C_{\epsilon_0} \subset [0, \epsilon_0] \equiv I_{\epsilon_0}^+$ be a Cantor set defined by an IFS of the form

$$f_1(t) = \lambda t, \quad f_2(t) = \lambda t - (\lambda/\epsilon_0 - 1)\epsilon_0 \quad (6)$$

where $\lambda = \beta\epsilon_0$, $0 < \beta < 1$ and $\alpha + 2\beta = 1$. Thus, at the first iteration an open interval O_{11} of size $\alpha\epsilon_0$ is removed from the interval $I_{\epsilon_0}^+$, at the second iteration 2 open intervals

O_{21} and O_{22} each of size $\alpha\epsilon_0(\beta)$ are removed and so on, so that a family of gaps O_{ij} of sizes $\alpha\epsilon_0(\beta)^{i-1}$, $j = 1, \dots, 2^{i-1}$ are removed in subsequent iterations from each of the closed subintervals I_{ij} , $j = 1, 2, \dots, 2^i$ of $I_{\epsilon_0}^+$. Consequently, $C_{\epsilon_0} = I_{\epsilon_0}^+ - \bigcup_i O_{ij} = \bigcap_i \bigcup_j I_{ij}$. Notice that the total length removed is $\sum \alpha\epsilon_0(2\beta)^{i-1} = \epsilon_0$, so that $m(C_{\epsilon_0}) = 0$.

Next, consider $\tilde{I}_N = [0, \beta^N]$ and let $N = n + r$ and $N \rightarrow \infty$ as $n \rightarrow \infty$ for a fixed $r \geq 0$. Choose the scale $\epsilon = \alpha\beta^n\epsilon_0$ and define $\tilde{t}_r \in [0, \alpha\beta^N\epsilon_0]$ a (positive) *relative infinitesimal* (relative to the scale ϵ) provided it also satisfies the *inversion* rule

$$\tilde{t}_r/\epsilon \propto \epsilon/t, \quad 0 < \tilde{t}_r < \epsilon < t \quad (7)$$

In the limit $\epsilon \rightarrow 0$, relative infinitesimals \tilde{t}_r , of course, vanish identically. However, the corresponding scale invariant infinitesimals $\tilde{\eta}_r = \tilde{t}_r/\epsilon$, $\epsilon \rightarrow 0$ may nevertheless be nontrivial. As a consequence, the relative infinitesimals may be awarded a new *scale invariant absolute value* [13, 14, 16]

$$v(\tilde{t}_r) \equiv |\tilde{t}_r|_u = \lim_{\epsilon \rightarrow 0} \log_{\epsilon^{-1}} \epsilon/|\tilde{t}_r|, \quad \tilde{t}_r \neq 0 \quad (8)$$

and $|0|_u = 0$. The set of infinitesimals are uncountable, and the above norm satisfies the stronger triangle inequality $|a + b|_u \leq \max\{|a|_u, |b|_u\}$. Accordingly, the zero set $\mathbf{0} = \{0, \pm\epsilon\tilde{\eta}_r \mid \tilde{\eta}_r \in (0, \beta^r), r = 0, 1, 2, \dots, \epsilon \rightarrow 0^+\}$ may be said to acquire *dynamically* the structure of a Cantor like ultrametric space, for each $\beta \in (0, 1/2)$ (so as to satisfy the open set condition [16]). The set $\mathbf{0}$ indeed is realized as a nested circles $S_r : \{\tilde{t} \mid |\tilde{t}_r|_u = \alpha_r\}$, in the ultrametric norm, when we order, without any loss of generality, $\alpha_0 > \alpha_1 > \dots$. For definiteness, we call 0 the hard (or stiff) zero, whereas elements of the asymptotic form $\pm\epsilon\tilde{\eta}_r$, $\epsilon \rightarrow 0^+$ are designated as soft (or dynamic) zeros.

To give a more pictorial representation of infinitesimals, let us recall that in a computational problem, a number $x = 1$, for example, is represented upto a finite accuracy; i.e., upto to a scale ϵ_0 , say. Then the numbers in the interval $(1 - \epsilon_0, 1 + \epsilon_0)$ are *computationally unobservable* and identified, as a whole, as the number 1. The above nontrivial construction now tells that the points in that computationally inaccessible limiting interval are *aligned dynamically* as nonintersecting clopen balls of a Cantor set C endowed with the nontrivial ultrametric value (8), thereby extending the ordinary set R to an infinite dimensional, scale free, nonarchimedean space \mathbf{R} accommodating *dynamically active* infinitesimals and infinities [c.f. Sec. 4]. A nonzero real variable $t(> 0$ say,) in R and approaching 0 now gets a deformed structure in \mathbf{R} of the form $T(t) = t.t^{-v(\tilde{t}(t))}$, thus reflecting a nontrivial effect of dynamic infinitesimals over the structure of the real number system R .

Next, we see that the above definition (8) is well defined and is not empty. For, let us assign $\phi(\tilde{t}) = v(\tilde{t}_r) \log \tilde{t}_r^{-1}$ a constant value for all $\tilde{\eta}_r \in (0, \beta^r)$. This constitutes a useful class of ultrametric norms when $\phi(\tilde{t})$ is identified with a *Cantor function* [16]. Indeed, for each arbitrarily small, but nonzero, $\epsilon \neq 0$, relative infinitesimals are elements of gaps of a Cantor set of the type C_ϵ , so that $\phi(\tilde{t}_r)$ is assigned constant values, in a piecewise sense, every where in $[0, \epsilon]$ so that $d\phi/dt = 0$ a.e in $[0, \epsilon]$. Indeed, the constant values

can always be assigned continuously so as to make it continuous and non-decreasing on $[0, \epsilon]$. However, at the end points of the gaps (that is, at the elements of C_ϵ), ϕ need not be differentiable in the usual sense. Hence, ϕ is uniquely realized as the Cantor function associated with C_ϵ .

In the present approach, however, ϕ is not only continuous, but also satisfies $\frac{d\phi}{dt} = 0$ every where in I_ϵ^+ (for details, see, for instance, [14, 16]). The usual derivative discontinuity is removed, since in the vicinity of $t_0 \in C_\epsilon$, transition between the end points of the gap containing t_0 is now accomplished smoothly by inversions of the form (7).

To summarise, the nontrivial infinitesimals are represented scale invariantly by the asymptotic formula of the form $\tilde{X}_\pm = X_\pm/x = x^{\mp v(\tilde{x})}$ as $x \rightarrow 0$ (from now on, t and x , as usual, denote time and space variables). Infinitesimals may therefore be interpreted as those numbers which approach 0 at a slower (nonlinear) rate: $\log \tilde{X}_-^{-1} = \log \tilde{X}_+ = v(\tilde{x}) \log \tilde{x}^{-1}$ as $x \rightarrow 0$. The solution (3) then provides a nontrivial representation for the fattened real number x , denoted as \mathbf{x} . The infinitesimals in a more conventional nonstandard model [15] of R are, however, inactive or passive in the present sense. The dynamical infinitesimals are already shown to have nontrivial influences in the asymptotic estimates of number theory and other areas of real analysis [14, 16]. Further studies will be reported elsewhere. For our latter reference, let us note here that one does enjoy a large body of resources of dynamical infinitesimals residing in a nontrivial neighbourhood of zero (0). Such an infinitesimal may be assumed to live in an infinitesimal copy of an arbitrarily assigned Cantor set. One has to make a *choice* of an appropriate Cantor set depending on the problem in hand. In the following we analyse the implications of the above nonarchimedean structure on a diffusion process on a measure zero Cantor set and also on \mathbf{R} .

2. Cantor set: New Results

The above ultrametric structure can easily be realized on a Cantor set (a compact and perfect subset of R). Recently, it is shown that the ultrametric so defined is both metrically and topologically inequivalent to the usual ultrametric that a Cantor set carries naturally [16]. For a point x_0 in a Cantor set $C \subset I = [a, b]$, the representation (3) now gives rise to a scale invariant ultrametric extension $\tilde{X}_\pm = X_\pm/x_0 = x_0^{\mp v(\tilde{x})}$ where the transition between two infinitesimally close scale invariant neighbours is supposed to be mediated by inversions of the form $\tilde{X} \rightarrow \tilde{X}^{-h}$ for a real h , which determines the jump size. Notice that \tilde{X} (and equivalently, $v(\tilde{x})$) is a locally constant Cantor like function and solves $\frac{d\tilde{X}}{dx} = 0$ everywhere in I . The ordinary discontinuity of a Cantor function at an $x_0 \in C$ is removed, since in the present ultrametric extension, the point x_0 in C is replaced by an *inverted Cantor set* which is the closure of *gaps* of an infinitesimal Cantor set C_i that is assumed to be the residence Cantor set for the relevant infinitesimals \tilde{x} living in the extended neighbourhood $\mathbf{0}$ of 0. The gaps of C_i constitute a disjoint family of connected clopen intervals (represented in a scale invariant manner) over each of which

scale invariant equations of the form (2) are well defined [16]. Consequently, the valuation $v(\tilde{x})$, redefined slightly in the modified form

$$(x/x_0)^{\tilde{v}(x)} = x_0^{v(\tilde{x})} \quad (9)$$

(that is, $\tilde{v}(x)/v(\tilde{x}) = \log x_0 / \log(x/x_0)$, exposing the relative variation of \tilde{v} over v), x assuming values from the gaps in the neighbourhood of x_0 , is realized as a smooth function defined recursively in a scale invariant way by the equation

$$\frac{d\tilde{v}(x)}{d\xi} = -\tilde{v}(x) \quad (10)$$

where $\xi = \log \log(x/x_0)$, $x \in C_i$. Recall that \tilde{x} resides in the gaps of nontrivial neighbourhood of 0 instead. As a consequence, \tilde{v} may be written as $\tilde{v}(x) = (\log(x/x_0))^k$, where k may be allowed to assume values from a set of scale factors related to that of the Cantor set. This form is clearly consistent with (9). Assuming x is drawn from a specific gap of a given size, the same, written more effectively as $\tilde{v}(x)/v(\tilde{x}) = (\log_{x_0}(x/x_0))^{-1}$, yields, in the limit of vanishingly small gaps (i.e., as $x \rightarrow x_0$ and vice versa), the limiting value $\tilde{v}_0(x)/v_0(\tilde{x}) = 1/s$, since $\lim \log_{x_0}(x/x_0) = s$ equals the finite Hausdorff dimension of the Cantor set C_i . Let us first note that if one replaces the Cantor set by a segment of a line of the form $(0, \delta)$, then $x/x_0 = 1$, in that limit ($\delta \rightarrow 0$) gives $s = 0$, which is consistent with the fact that the line segment reduces to a point, viz. 0. In the general case, $x/x_0 \propto N$, the number of clopen balls that covers the fattened gap of the form $(x, x_0) \subset (0, \delta)$ (size of balls are determined by the gap). Letting $x_0 \rightarrow 0$ (as the relevant scale factors β^n), the above limit therefore mimics the box dimension, which also equals the Hausdorff dimension of the Cantor set concerned. The topological inequivalence of the present ultrametric arises from the possible dichotomy in the choice of C_i .

Let us remark that the ordinary limit $\epsilon \rightarrow 0$ in R is altered because of scale invariant dynamic infinitesimals $\log \tilde{x}/\epsilon \approx v(\tilde{x}) \log \epsilon^{-1} \approx \epsilon \log \epsilon^{-1}$, when \tilde{x} is considered to lie on a fattened (connected) gap, so that the ultrametric valuation may be assumed to coincide with the usual (Euclidean) value viz. $v(\tilde{x}) \approx \epsilon$. However, assuming $\tilde{\epsilon}(= \beta^n, n \rightarrow \infty)$ to be an infinitesimal scale of the Cantor set concerned, we also have $\log \tilde{x}/\tilde{\epsilon} \approx \tilde{\epsilon}^s \log \tilde{\epsilon}^{-1}$, since the valuation is identified with the associated Cantor function $\phi(\tilde{x}) \approx \tilde{\epsilon}^s \approx \epsilon$, s being, as usual, the corresponding Hausdorff dimension. Reverting back to the ordinary scale ϵ (and keeping in view the associated scale invariance), this scaling can be identified with $\epsilon^{\tilde{s}} \approx O(1)\epsilon \log \epsilon^{-1}$, for an \tilde{s} given by $\tilde{s} \approx 1 - \frac{\log \log \epsilon^{-1}}{\log \epsilon^{-1}}$. As a consequence, in the presence of an ultrametric space, the ordinary limit $\epsilon \rightarrow 0$ is replaced by the sublinear limit

$$\epsilon^{\tilde{s}} = \epsilon \log \epsilon^{-1} \rightarrow 0, \quad (11)$$

$0 < \tilde{s} < 1$, as $\epsilon \rightarrow 0$. This is one of the main results of this paper. A real variable t in R approaching to (or flowing out from) 0 will experience this scale invariant sublinear behaviour in an incredibly small neighbourhood of **0** in **R** and should have a deep significance in number theory and other areas [14]. Similar behaviour is also reported recently

in the context of diffusion in an ultrametric Cantor set in a noncommutative space [8]. Finally, the ultrametric induced by the valuation $v(\tilde{x})$ coincides with the natural ultrametric only when the scaling properties of C_i coincide with that of C . In this paper we adhere to the latter possibility.

Suppose the original Cantor set C and the infinitesimal Cantor set C_i have Hausdorff dimensions s and s' respectively. Any point \mathbf{x} of the fattened set $\mathbf{C} = C + C_i$ is given as $\mathbf{x} = x + \tilde{x}$, $x \in C$, $\tilde{x} \in C_i$. It is well known that $\mathbf{C} = I$, for a.e. s' , for a given s [19]. Accordingly, it follows that given a Lebesgue measure zero Cantor set C , the above smooth differentiable structure is a.s realized on \mathbf{C} , which is nothing but I , though in an appropriate (scale free) logarithmic variable.

To understand more clearly the above smooth scale invariant structure let us consider the classical middle third Cantor set $C_{1/3}$ with scale factors $\tilde{\epsilon} = 3^{-n}$. A point $x_0 = 3^{-n} \sum a_i 3^{-i}$, $a_i \in \{0, 2\}$ of $C_{1/3}$ is raised to the scale free \mathbf{x} which is a variable living in a family of fattened gaps, attached and structured hierarchically at the point x_0 (or equivalently, by scale invariance, at 1), over each of which scale free equations of the form (10) are valid. The infinitesimals are elements of the gaps “closest” to 0, viz. the open intervals $3^{-n-m}(1, 2)$, in the limit $n \rightarrow \infty$, for a fixed $0 < m < n$, which are assigned nontrivial values akin to the Cantor function $v(\tilde{x}) = i3^{-ms}$, $i = 1, 2, \dots, 2^m - 1$ [13]. Over each of the finite size gaps, on the otherhand, the valuation $v(x)$ is awarded as $v(x) = 3^{-sn}$. Both these valuations are not only continuous but also smooth since the corresponding Cantor function is realized as a smooth function via the logarithmic ansatz for a substitution of the form $3^n \Delta x_n = 3^n(x - x_n) \rightarrow n \log \frac{\mathbf{x}}{x_n}$, as $n \rightarrow \infty$, thereby removing the derivative discontinuity at the points of scale changes [13], so that $\frac{dv(x)}{dx} = 0$, every where on the Cantor set concerned. Notice that gaps scale as $\epsilon = 2^{-n}$ (recall the binary representation for points on a connected segment of the real line) when a closed interval containing points like $x_0 \in C_{1/3}$ scales as 3^{-n} , so that the Hausdorff dimension is $s = \log_3 2$. By (9), the variability of the valuation $\tilde{v}(x)$ in the limit of vanishing gap sizes is obtained as $\tilde{v}_0(x) \propto 3^{-sn} s^{-1}$, $n \rightarrow \infty$.

Next, before determining the incremental *measure*, denoted $d_j \tilde{X}$, of smooth self similar jump processes of (gap) “size” (in the sense of a weight) ϵ (2^{-n} , for $C_{1/3}$, say) in the neighbourhood of the scale invariant 1, let us first recall that pure translations follow a linear law: $y = Tx = x + h$. The *instantaneous pure jumps* (of unit length close to the scale invariant 1), on the other hand, follow a *hyperbolic law*: $\tilde{X} \rightarrow Y = \tilde{X}^{-1} \Rightarrow \log Y + \log X = 0$, which tells, in turn, that the corresponding translational increment, even in the log scale, is indeed zero. This actually is the case for the valuation (3) defined in terms of the locally constant Cantor function. The (manifestly scale invariant) multiplicative valuation $\tilde{v}(x) = \log_{x^{-1}}(X/x)$ in (3), however, gives the correct linear measure for a single jump relative to the point x (for the above hyperbolic type jump, $v(x) = 1$ relative to x itself as the scale). The corresponding *multiplicative increment* is denoted as $\delta_j \tilde{X} = (x/x_0)^{\tilde{v}(x)}$. More importantly, this valuation is realized as a smooth measure and may be considered to contribute an independent component in the ordinary measure of R . Further, the total

self similar jump mediated increments over a spectrum of gaps of various sizes of the forms $\epsilon_n = 2^{-n}\epsilon_0$ in the neighbourhood of a (middle third) Cantor point x_0 (say) is now obtained as

$$\Delta_j \tilde{X} = (x/x_0)^{s^{-1}2^{-m}\sum_n 2^{-n}} \quad (12)$$

which in the limit $x \rightarrow x_0$, that is, $m \rightarrow \infty$ yields the *jump differential* $d_j \tilde{X} = (d\tilde{x})^{s^{-1}}$, where $\tilde{x} = \lim(x/x_0)^{2^{-m}}$, is a deformed variable close to 1. Such a variable ($\neq 1$ exactly) exists because of a nontrivial g.l.b of gap sizes (another manifestation of the sublinear asymptotic). Clearly, the jump differential appears to relate to a fractional differential of order s^{-1} . Invoking further the inherent smoothness of the formalism we, however, treat this fractional differential as the ordinary differential $d_j \tilde{X} = d\tilde{x}^{s^{-1}}$. One justification, at least huerisitically, is the following. The jump process, although realized here as smooth, carries an inherent random element [12]. The jump measure must therefore be interpreted as an equality of moments $\langle d\tilde{x} \rangle^{s^{-1}} \approx d \langle \tilde{x}^{s^{-1}} \rangle$. A possible distribution with analogous moment constraints may be Poisson like but not exactly Poissonian, an example of which is considered in [17]. Incidentally, we note that the essential singularity in $s = 0$ tells that in the absence of inversion mediated jumps, the whole structure of gaps collapses to a point (singleton set, devoid of any nontrivial infinitesimals). The divergence in the jump measure then reflects the ordinary nondifferentiable structure of the Cantor set. On the otherhand, on any connected segment of R , $s = 1$, and the jump measure reduces to the ordinary linear measure dx . To summarise, *the significantly new insight that emerges from the above analysis is that an infinitesimal scale invariant increment on an ultrametric space must have the form $\tilde{X} = 1 + \epsilon^{1/s}$, $\epsilon \rightarrow 0^+$ on a connected segment close to 0*. Recalling $\tilde{\epsilon} = \epsilon^{1/s}$, the above infinitesimal jump increment $\tilde{X} = 1 \pm \tilde{\epsilon}$ reduces to the usual increment on a Cantor set in the usual metric, but at the cost of the smooth structure.

3. Diffusion

Coming back to the diffusion equation (1), let us next recall the self similarity of the same. Writing $W(x, t) = t^{-1/2}w(z)$, $z = \frac{x}{\sqrt{t}}$, the scaling function w satisfies the first order ordinary differential equation

$$\frac{dw}{du} = -w, \quad u = z^2 \quad (13)$$

giving rise to the gaussian propagator $W(x, t) = At^{-1/2}e^{-\frac{x^2}{4t}}$ by a direct integration.

Now, in the present analysis, the real variables x and $t > 0$ must be assumed to live in the corresponding scale invariant extensions \mathbf{R} and \mathbf{R}_+ (set of nonnegative numbers), so that the scaling variable z gets extended to a deformed variable $\tilde{Z} = z/z_0 = \frac{\tilde{X}}{\sqrt{\tilde{T}}} \in \mathbf{1}$ in the extended neighbourhood of a point $(x_0, t_0) \in R \times R_+$. The scale invariant variables \tilde{X} and \tilde{T} , in turn, belong to two Cantor sets C_s and C_t respectively with scale invariant measures

$d_j \tilde{X} = d\tilde{x}^\alpha$ and $d_j \tilde{T} = d\tilde{t}^\beta$, where α and β are the respective inverse Hausdorff dimensions and \tilde{x} and \tilde{t} are two scale invariant variables near 1 of R . As a consequence, Eq(13), defined originally on R , now automatically gets extended to one in the new rescaling symmetric variable \tilde{Z} living on a ultrametric Cantor set in the extended neighbourhood of every point $(x, t) \in R \times R_+$ and hence supports nontrivial solutions analogous to (3). Thus, integrating (13) in that class of new solutions, we get

$$W_C(\tilde{X}, \tilde{T}) = A\tilde{t}^{-\beta/2} e^{-\lambda(\frac{\tilde{x}^{2\alpha}}{\tilde{t}^\beta})^{1+\nu}} \quad (14)$$

as a *stretched exponential* fat tailed propagator for a diffusive process (random walker) on a Cantor set and λ is a constant depending on z_0 . Clearly, W_C satisfies the scale invariant diffusion equation

$$\frac{\partial W}{\partial \tilde{T}} = \frac{\partial^2 W}{\partial \tilde{X}^2} \quad (15)$$

which is defined close to every point $x \in C_s$ and at any instant $t \in C_t$. Although the derivatives are evaluated with jump differentials, these are equivalent to the usual partial derivatives, but in the deformed (scaling) variables x^α and t^β respectively. Further, the exponent ν in (14) is a valuation so that $\tilde{Z}^{\nu(\tilde{Z})}$ is a locally constant Cantor function that arises in connection with the residence Cantor set for the variable z (c.f. eq(3)) [20]. Because of the sublinear asymptotic increments of the form (11), this equation is also considered to be valid on a connected line segment close to $t = 0$, so that (15) is also valid for $\tilde{t} \approx 0$. Next, we note that the scale invariant factors of a real variable (viz. eq(3)) become significant only for an asymptotically large time. Consequently, the above scale invariant solution (14) of the diffusion equation (1) is expected to arise naturally in any diffusive process that persists over many longer time scales living in a set of the form **R**. The anomalous mean square deviation is given generically as $\langle \Delta x^2(t) \rangle = t^{\beta/\alpha}$, where $t \rightarrow 0^+$, for every scale invariant x near 1.

We conclude that a simple diffusion process if allowed to evolve over many longer and longer time scales as those available for natural processes will ultimately give away naturally to a *stretched exponential* nongaussian distribution of the form (14) leading to an anomalous mean square fluctuation; reflecting, in turn, the universally present scale invariant numerical fluctuations [21].

Diffusion on a Cantor set when examined in the framework of the ordinary Newtonian time (i.e., when C_t reduces to the singleton $\{0\}$) is generally subdiffusive with exponent $\alpha^{-1} = s : 0 < s < 1$, s being the Hausdorff dimension of the diffusive medium (since $\beta = 1$). The mean square deviation has the generic form $\langle \Delta x^2(t) \rangle = t^s \approx t \log t^{-1}$, $t \rightarrow 0^+$, because of the sublinear asymptotic flow on a Cantor set. As already remarked, because of scale invariance this may also be interpreted as the asymptotic late time anomalous mean square fluctuation from the gaussian law even for the ordinary diffusion equation (1).

For a fractal time process the subdiffusion occurs for $s < \tilde{s}$ and superdiffusion for $s > \tilde{s}$, \tilde{s} being the Hausdorff dimension for underlying Cantor like set for the fractal time.

However, for $s = \tilde{s}$, the gaussian like linear mean square variation may be observed even for a fractal time process [4]. Analogous smoothening in the asymptotic scaling of the eigen value counting function was also noticed by Freiberg [7].

4. Final Remarks

Before closing, let us note that the usual “cut and delete” process realizing a Cantor set seems to give a misleading idea that the Lebesgue measure of a set, for instance, $[0,1]$ becomes zero under a recursive application of a contraction mapping of the form $f(x) = x/3$. However, the full Lebesgue measure could actually be preserved multiplicatively at every level: $1 = 3^{-n} \times q^{ns}$ for a given $q > 0$, leading to a Cantor set of dimension $s = \frac{\log 3}{\log q}$, that remains attached to the limit point of the said contraction. It is reasonable to imagine that the (static) points of the interval $[0,1]$ dynamically adjust among themselves in a *scrambled* manner to accommodate around the limit point a Cantor set with fractal dimension s . The intrinsic motion of these dynamic numbers (in the logarithmic scale) could be visualized as an anomalous Brownian motion given by (15). Nature seems to make use in plenty this seemingly universal principle of *making space out of nothing!*

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